Homework 2

Deadline: May 11, 2023

1. With the notations in Section 2.3, let $\mathbf{b} = \mathbf{0}$ and

$$\mathbf{A} = \begin{pmatrix} 7 & 3 & 2 \\ 3 & 4 & 1 \\ 2 & 1 & 2 \end{pmatrix},$$

   (a) Write the density.

   (b) Find $\mathbf{\Sigma}$.

2. Let $\mu = \mathbf{0}$ and

$$\mathbf{\Sigma} = \begin{pmatrix} 1 & 0.8 & -0.4 \\ 0.8 & 1 & -0.56 \\ -0.4 & -0.56 & 1.0 \end{pmatrix}$$

   (a) Find the conditional distribution of X_1 and X_3, given $X_2 = x_2$.

   (b) What is the partial correlation between X_1 and X_3 given X_2.

3. Let $\mathbf{X}$ be distributed according to $N(\boldsymbol{\mu}, \mathbf{\Sigma})$ with $\boldsymbol{\mu} = \mathbf{0}$. Differentiating the characteristic function, verify that

   (a) $\mathbb{E}(X_i - \mu_i)(X_j - \mu_j)(X_k - \mu_k) = 0$,

   (b) $\mathbb{E}(X_i - \mu_i)(X_j - \mu_j)(X_k - \mu_k)(X_l - \mu_l) = \sigma_{ij}\sigma_{kl} + \sigma_{ik}\sigma_{jl} + \sigma_{il}\sigma_{jk}$.

4. Show that when $\mathbf{X}$ is normally distributed, the components are mutually independent if and only if the covariance matrix is diagonal.

5. *Estimation of $\mathbf{\Sigma}$ when $\boldsymbol{\mu}$ is known.* Show that if $\mathbf{x}_1, \ldots, \mathbf{x}_N$ constitute a sample from $N(\boldsymbol{\mu}, \mathbf{\Sigma})$ and $\boldsymbol{\mu}$ is known, then

$$\frac{1}{N}\sum_{\alpha=1}^{N}(\mathbf{x}_\alpha - \boldsymbol{\mu})(\mathbf{x}_\alpha - \boldsymbol{\mu})^\top$$

   is the maximum likelihood estimator of $\mathbf{\Sigma}$.