Homework 1

Deadline: April 9, 2023

1. For any $\mathbf{A} \in \mathbb{S}^n$, $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^n$, prove that $\mathbf{x}^\top \mathbf{A}\mathbf{y} = \mathbf{y}^\top \mathbf{A}\mathbf{x}$.

2. Prove that for any matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$, the column rank of $\mathbf{A}$ is equal to the row rank of $\mathbf{A}$.

3. For $\mathbf{A} \in \mathbb{R}^{m\times n}$ and $\mathbf{B} \in \mathbb{R}^{n\times p}$, prove that $\|\mathbf{A}\mathbf{B}\|_F \le \|\mathbf{A}\|_F \|\mathbf{B}\|_F$.

4. Suppose $\mathbf{A} \in \mathbb{R}^{n\times n}$ has eigenvalues $\lambda_1, \ldots, \lambda_n$. Prove the following statements

   (a) $\mathrm{tr}(\mathbf{A}) = \sum_{i=1}^n \lambda_i$,

   (b) $\det(\mathbf{A}) = \prod_{i=1}^n \lambda_i$.

5. Prove the SVD always exists for any $\mathbf{A} \in \mathbb{R}^{m\times n}$. (Hint: Using spectral decomposition theorem)

6. Given the symmetric matrix

$$\mathbf{N} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^\top & \mathbf{D} \end{bmatrix}$$

   with non-singular $\mathbf{A}$ and let $\mathbf{S} = \mathbf{D} - \mathbf{B}^\top \mathbf{A}^{-1}\mathbf{B}$. Prove that

   (a) $\mathbf{N} \succ \mathbf{0} \iff \mathbf{A} \succ \mathbf{0}$ and $\mathbf{S} \succ \mathbf{0}$.

   (b) If $\mathbf{A} \succ \mathbf{0}$, then $\mathbf{N} \succeq \mathbf{0} \iff \mathbf{S} \succeq \mathbf{0}$.